Find two linearly independent series solutions of $2x^2y''+(2x^2+3x)y'-y=0$ about $x=0$. **SCORE: _____ / 17 PTS**

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+(2x^2+3x)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

① POINT EACH EXCEPT AS NOTED

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}2(n+r)a_nx^{n+r+1}+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=1}^{\infty}2(n+r-1)a_{n-1}x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$\frac{1}{2}$

$$2r(r-1)a_0x^r+3ra_0x^r-a_0x^r+\sum_{n=1}^{\infty}[(2(n+r)(n+r-1)+3(n+r)-1)a_n+2(n+r-1)a_{n-1}]x^{n+r}=0$$

$$(2r(r-1)+3r-1)a_0x^r+\sum_{n=1}^{\infty}[(2(n+r)(n+r-1)+3(n+r)-1)a_n+2(n+r-1)a_{n-1}]x^{n+r}=0$$

$$2r(r-1)+3r-1=0 \;\Rightarrow\; 2r^2+r-1=0 \;\Rightarrow\; (2r-1)(r+1)=0 \;\Rightarrow\; r=\tfrac{1}{2},-1$$

$$(2(n+r)(n+r-1)+3(n+r)-1)a_n+2(n+r-1)a_{n-1}=0 \;\Rightarrow$$

$$a_n=-\frac{2(n+r-1)}{2(n+r)(n+r-1)+3(n+r)-1}a_{n-1}=-\frac{2(n+r-1)}{(2(n+r)-1)(n+r+1)}a_{n-1}=-\frac{2(n+r-1)}{(2n+2r-1)(n+r+1)}a_{n-1}$$

$r=\frac{1}{2} \;\Rightarrow\; a_n=-\frac{2(n-\frac{1}{2})}{2n(n+\frac{3}{2})}a_{n-1}=-\frac{2n-1}{n(2n+3)}a_{n-1}$

$a_0=1$

$a_1=-\frac{(1)}{(1)(5)}a_0=-\frac{1}{1\cdot5}$

$a_2=-\frac{(3)}{(2)(7)}a_1=\frac{1\cdot3}{(1\cdot2)(5\cdot7)}$

$a_3=-\frac{(5)}{(3)(9)}a_2=-\frac{1\cdot3\cdot5}{(1\cdot2\cdot3)(5\cdot7\cdot9)}$

$y_1=x^{\frac{1}{2}}(1-\frac{1}{1\cdot5}x+\frac{1\cdot3}{(1\cdot2)(5\cdot7)}x^2-\frac{1\cdot3\cdot5}{(1\cdot2\cdot3)(5\cdot7\cdot9)}x^3+\cdots)$

$y_1=x^{\frac{1}{2}}(1-\frac{1}{5}x+\frac{3}{70}x^2-\frac{1}{126}x^3+\cdots)$

④

$r=-1 \;\Rightarrow\; a_n=-\frac{2(n-2)}{(2n-3)n}a_{n-1}$

$a_0=1$

$a_1=-\frac{2(-1)}{(-1)(1)}a_0=-2$

$a_2=-\frac{2(0)}{(1)(2)}a_1=0=a_3=a_4=a_5=\cdots$

$y_2=x^{-1}(1-2x)=\frac{1}{x}-2$

②

Find two linearly independent series solutions of $(2x-1)y''-xy'+2y=0$ about $x=0$. **SCORE: _____ / 13 PTS**

$$(2x-1)\boxed{\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}}-x\boxed{\sum_{n=1}^{\infty}na_nx^{n-1}}+2\boxed{\sum_{n=0}^{\infty}a_nx^n}=0$$

① POINT EACH EXCEPT AS NOTED

$$\overset{\frac{1}{2}}{\boxed{\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-1}}}-\overset{\frac{1}{2}}{\boxed{\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}}}-\overset{\frac{1}{2}}{\boxed{\sum_{n=1}^{\infty}na_nx^n}}+\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\overset{\frac{1}{2}}{\boxed{\sum_{n=1}^{\infty}2(n+1)na_{n+1}x^n}}-\overset{\frac{1}{2}}{\boxed{\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n}}-\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\boxed{-2a_2+2a_0}+\boxed{\sum_{n=1}^{\infty}[2(n+1)na_{n+1}-(n+2)(n+1)a_{n+2}-(n-2)a_n]x^n}=0$$

$$-2a_2+2a_0=0\ \Rightarrow\ \boxed{a_2=a_0}\ \tfrac{1}{2}$$

$$2(n+1)na_{n+1}-(n+2)(n+1)a_{n+2}-(n-2)a_n=0\ \Rightarrow\ \boxed{a_{n+2}=\frac{2n(n+1)a_{n+1}-(n-2)a_n}{(n+2)(n+1)}}\ \text{ for } n\in Z^+$$

$a_0=1,\quad a_1=0$

$a_2=a_0=1$

$a_3=\frac{2(1)(2)a_2-(-1)a_1}{(3)(2)}=\frac{4}{(3)(2)}=\frac{2}{3}$

$a_4=\frac{2(2)(3)a_3-(0)a_2}{(4)(3)}=\frac{8}{(4)(3)}=\frac{2}{3}$

$y_1=\boxed{1+x^2+\frac{2}{3}x^3+\frac{2}{3}x^4}+\cdots$

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$a_0=0,\quad a_1=1$

$a_2=0$

$a_3=\frac{2(1)(2)a_2-(-1)a_1}{(3)(2)}=\frac{1}{(3)(2)}=\frac{1}{6}$

$a_4=\frac{2(2)(3)a_3-(0)a_2}{(4)(3)}=\frac{2}{(4)(3)}=\frac{1}{6}$

$a_5=\frac{2(3)(4)a_4-(1)a_3}{(5)(4)}=\frac{\frac{23}{6}}{(5)(4)}=\frac{23}{120}$

$y_1=\boxed{x+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{23}{120}x^5}+\cdots$

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